APMA 1650 – Homework 4

Due Monday, July 18, 2016

Homework is due during class or by 3:45 pm in the homework drop box in 182 George St. Show all of your work used in deriving your solutions.

1. Let X be a continuous random variable which has a probability density $f(x)$ given by

$$f(x) = \begin{cases} cx^3 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Find the value of c which makes this a valid probability density function.

   (b) What is $\mathbb{P}(0 \leq X \leq \frac{1}{4})$?

   (c) Find the expected value and variance of X.

   (d) Find the median of X.

2. Suppose we are playing a game of darts. We throw darts at a circular dartboard with radius 1. Suppose the darts land uniformly at random on the dartboard, i.e. the probability of landing in a subset of the dartboard is equal to the area of the subset divided by the area of the dartboard.

   (a) What is the probability that the dart hits the exact center of the dartboard?

   (b) What is the probability that the dart lands closer to the center than to the edge of the dartboard?

   (c) For $a < b < 1$, what is the probability that the dart lies at a distance between a and b of the center of the dartboard?

3. A company that manufactures and bottles juice uses a machine that automatically fills 16-ounce bottles. There is some variation in the amount of liquid dispensed by the machine. The amount of juice dispensed has been observed to follow a normal distribution with mean of 16 ounces and standard deviation of 0.6 ounces.

   (a) What is the probability that the machine dispenses more than 17 ounces?

   (b) What is the probability that the machine dispenses between 15.5 and 16.5 ounces?

   (c) Suppose you wish to be 95% confident that the machine dispenses between 15.5 and 16.5 ounces. What standard deviation must your machine have for this to be the case?

4. Yet another time, you find yourself the quality control manager for the Acme Widget Company. You are concerned about the number of defective widgets being produced by one of your factories.

(a) You have determined that the number of defective widgets produced per day by your factory has a mean of 10. You claim that the probability that your factory produces 15 or more defective widgets per day is less than or equal to 0.5. Is this claim justified? Explain your answer mathematically.

(b) In addition, you have determined that the number of defective widgets produced per day by your factory has a variance of 5. You claim that the probability that your factory produces 15 or more defective widgets per day is less than or equal to 0.1. Is this claim justified? Explain your answer mathematically.

(c) In addition, you have determined that the number of defective widgets produced per day by your factory has a distribution which is symmetric about its mean. You claim that the probability that your factory produces 15 or more defective widgets per day is less than or equal to 0.1. Is this claim justified? Explain your answer mathematically.

(d) Finally, you have have determined that the number of defective widgets produced per day by your factory is approximately a normal distribution (with the mean and variance above). You claim that the probability that your factory produces 15 or more defective widgets per day is less than or equal to 0.02. Is this claim justified? Explain your answer mathematically.

5. In the previous homework we gave an example of approximating a binomial distribution by a Poisson distribution. In this problem, we will approximate a binomial distribution by a normal distribution. From looking at the pmfs of the binomial distribution (in class and in the notes), for large values of n the histogram of the binomial pmf looks "bell-shaped". It turns out that the normal distribution is a good approximation for the binomial approximation for large n. It also works for small n as long as p is not too far from 0.5. For convenience, we will let $q = 1 - p$.

(a) Let $X \sim \text{Binomial}(25, 0.4)$. Compute $\mathbb{P}(X = 10)$.

(b) We will approximate X with a normal random variable. Let Y be a normal random variable with the same mean and variance as X. Compute $\mathbb{P}(9.5 \leq Y \leq 10.5)$. We will use this as an approximation for $\mathbb{P}(X = 10)$.

(c) Briefly explain why we used the event $(9.5 \leq Y \leq 10.5)$ to approximate the event $(X = 10)$.

(d) Compute the relative error in your approximation. You may refer back to the previous problem set for the definition of relative error.

The general rule is that the normal approximation is "good enough" if

$$0 < p \pm 3\sqrt{\frac{pq}{n}} < 1$$

(e) Show that $0 \leq p \pm 3\sqrt{\frac{pq}{n}} \leq 1$ implies that

$$n \geq 9\left(\frac{p}{q}\right) \text{ and } n \geq 9\left(\frac{q}{p}\right)$$

(f) Using this rule, how large should n be to approximate a binomial distribution with $p = 0.4$, $p = 0.8$, $p = 0.9$, and $p = 0.99$? Was our approximation in part (b) justified according to this rule?